6.1 Introduction

Many practical systems are sufficiently nonlinear so that the important features of their performance may be completely overlooked if they are analyzed and designed through linear techniques. The mathematical models of the nonlinear systems are represented by nonlinear differential equations. Hence, there are no general methods for the analysis and synthesis of nonlinear control systems. The fact that superposition principle does not apply to nonlinear systems makes generalisation difficult and study of many nonlinear systems has to be specific for typical situations.

6.2 Behaviour of Nonlinear Systems

The most important feature of nonlinear systems is that nonlinear systems do not obey the principle of superposition. Due to this reason, in contrast to the linear case, the response of nonlinear systems to a particular test signal is no guide to their behaviour to other inputs. The nonlinear system response may be highly sensitive to input amplitude. For example, a nonlinear system giving best response for a certain step input may exhibit highly unsatisfactory behaviour when the input amplitude is changed. Hence, in a nonlinear system, the stability is very much dependent on the input and also the initial state.

Further, the nonlinear systems may exhibit limit cycles which are self-sustained oscillations of fixed frequency and amplitude. Once the system trajectories converge to a limit cycle, it will continue to remain in the closed trajectory in the state space identified as limit cycles. In many systems the limit cycles are undesirable particularly when the amplitude is not small and result in some unwanted phenomena.

A nonlinear system, when excited by a sinusoidal input, may generate several harmonics in addition to the fundamental corresponding to the input frequency. The amplitude of the fundamental is usually the largest, but the harmonics may be of significant amplitude in many situations.

Another peculiar characteristic exhibited by nonlinear systems is called jump phenomenon. For example, let us consider the frequency response curve of spring-mass-damper system. The frequency responses of the system with a linear spring, hard spring and soft spring are as shown in Fig. 6.2(a), Fig. 6.2(b) and Fig. 6.2(c) respectively. For a hard spring, as the input frequency is gradually increased from zero, the measured
response follows the curve through the A, B and C, but at C an increment in frequency results in discontinuous jump down to the point D, after which with further increase in frequency, the response curve follows through DE. If the frequency is now decreased, the response follows the curve EDF with a jump up to B from the point F and then the response curve moves towards A. This phenomenon which is peculiar to nonlinear systems is known as jump resonance. For a soft spring, jump phenomenon will happen as shown in fig. 6.2(c).

![Fig. 6.2(a)](image1) ![Fig. 6.2(b)](image2) ![Fig. 6.2(c)](image3)

When excited by a sinusoidal input of constant frequency and the amplitude is increased from low values, the output frequency at some point exactly matches with the input frequency and continue to remain as such thereafter. This phenomenon which results in a synchronisation or matching of the output frequency with the input frequency is called frequency entrainment or synchronisation.

### 6.3 Methods of Analysis

Nonlinear systems are difficult to analyse and arriving at general conclusions are tedious. However, starting with the classical techniques for the solution of standard nonlinear differential equations, several techniques have been evolved which suit different types of analysis. It should be emphasised that very often the conclusions arrived at will be useful for the system under specified conditions and do not always lead to generalisations. The commonly used methods are listed below.

**Linearization Techniques:** In reality all systems are nonlinear and linear systems are only approximations of the nonlinear systems. In some cases, the linearization yields useful information whereas in some other cases, linearised model has to be modified when the operating point moves from one to another. Many techniques like perturbation method, series approximation techniques, quasi-linearization techniques etc. are used for linearise a nonlinear system.

**Phase Plane Analysis:** This method is applicable to second order linear or nonlinear systems for the study of the nature of phase trajectories near the equilibrium points. The system behaviour is qualitatively analysed along with design of system parameters so as to get the desired response from the system. The periodic oscillations in nonlinear systems called limit cycle can be identified with this method which helps in investigating the stability of the system.

**Describing Function Analysis:** This method is based on the principle of harmonic linearization in which for certain class of nonlinear systems with low pass characteristic. This method is useful for the study of existence of limit cycles and determination of the amplitude, frequency and stability of these limit cycles. Accuracy is better for higher order systems as they have better low pass characteristic.
**Liapunov’s Method for Stability:** The analytical solution of a nonlinear system is rarely possible. If a numerical solution is attempted, the question of stability behaviour can not be fully answered as solutions to an infinite set of initial conditions are needed. The Russian mathematician A.M. Liapunov introduced and formalised a method which allows one to conclude about the stability without solving the system equations.

### 6.4 Classification of Nonlinearities

The nonlinearities are classified into i) Inherent nonlinearities and ii) Intentional nonlinearities.

The nonlinearities which are present in the components used in system due to the inherent imperfections or properties of the system are known as inherent nonlinearities. Examples are saturation in magnetic circuits, dead zone, back lash in gears etc. However in some cases introduction of nonlinearity may improve the performance of the system, make the system more economical consuming less space and more reliable than the linear system designed to achieve the same objective. Such nonlinearities introduced intentionally to improve the system performance are known as intentional nonlinearities. Examples are different types of relays which are very frequently used to perform various tasks. But it should be noted that the improvement in system performance due to nonlinearity is possible only under specific operating conditions. For other conditions, generally nonlinearity degrades the performance of the system.

### 6.5 Common Physical Nonlinearities:

The common examples of physical nonlinearities are saturation, dead zone, coulomb friction, stiction, backlash, different types of springs, different types of relays etc.

**Saturation:** This is the most common of all nonlinearities. All practical systems, when driven by sufficiently large signals, exhibit the phenomenon of saturation due to limitations of physical capabilities of their components. Saturation is a common phenomenon in magnetic circuits and amplifiers.

**Dead zone:** Some systems do not respond to very small input signals. For a particular range of input, the output is zero. This is called dead zone existing in a system. The input-output curve is shown in figure.

![Saturation and Dead Zone](image)

Figure 6.3

**Backlash:** Another important nonlinearity commonly occurring in physical systems is hysteresis in mechanical transmission such as gear trains and linkages. This nonlinearity is somewhat different from magnetic hysteresis and is commonly referred to as backlash. In servo systems, the gear backlash may cause sustained oscillations or chattering phenomenon and the system may even turn unstable for large backlash.
Figure 6.4

Relay: A relay is a nonlinear power amplifier which can provide large power amplification inexpensively and is therefore deliberately introduced in control systems. A relay controlled system can be switched abruptly between several discrete states which are usually off, full forward and full reverse. Relay controlled systems find wide applications in the control field. The characteristic of an ideal relay is as shown in figure. In practice a relay has a definite amount of dead zone as shown. This dead zone is caused by the facts that relay coil requires a finite amount of current to actuate the relay. Further, since a larger coil current is needed to close the relay than the current at which the relay drops out, the characteristic always exhibits hysteresis.

Multivariable Nonlinearity: Some nonlinearities such as the torque-speed characteristics of a servomotor, transistor characteristics etc., are functions of more than one variable. Such nonlinearities are called multivariable nonlinearities.
Chapter 7

Phase Plane Analysis

7.1 Introduction

Phase plane analysis is one of the earliest techniques developed for the study of second order nonlinear systems. It may be noted that in the state space formulation, the state variables chosen are usually the output and its derivatives. The phase plane is thus a state plane where the two state variables \( x_1 \) and \( x_2 \) are analysed which may be the output variable \( y \) and its derivative \( \dot{y} \). The method was first introduced by 
Poincare, a French mathematician. The method is used for obtaining graphically a solution of the following two simultaneous equations of an autonomous system.

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*}
\]

Where \( \dot{x}_1 = f_1(x_1, x_2) \) and \( \dot{x}_2 = f_2(x_1, x_2) \) are either linear or nonlinear functions of the state variables \( x_1 \) and \( x_2 \) respectively. The state plane with coordinate axes \( x_1 \) and \( x_2 \) is called the phase plane. In many cases, particularly in the phase variable representation of systems, take the form

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*}
\]

The plot of the state trajectories or phase trajectories of above said equation thus gives an idea of the solution of the state as time \( t \) evolves without explicitly solving for the state. The phase plane analysis is particularly suited to second order nonlinear systems with no input or constant inputs. It can be extended to cover other inputs as well such as ramp inputs, pulse inputs and impulse inputs.

7.2 Phase Portraits

From the fundamental theorem of uniqueness of solutions of the state equations or differential equations, it can be seen that the solution of the state equation starting from an initial state in the state space is unique. This will be true if \( f_1(x_1, x_2) \) and \( f_2(x_1, x_2) \) are analytic. For such a system, consider the points in the state space at which the derivatives of all the state variables are zero. These points are called singular points. These are in fact equilibrium points of the system. If the system is placed at such a point, it will continue to lie there if left undisturbed. A family of phase trajectories starting from different initial states is called a phase portrait. As time \( t \) increases, the phase portrait graphically shows how the system moves in the entire state plane from the initial states in the different regions. Since the solutions from each of the initial conditions are unique, the phase trajectories do not cross one another. If the system has nonlinear elements which are piece-wise linear, the complete state space can be divided into different regions and phase plane trajectories constructed for each of the regions separately.

7.3 Phase Plane Method

Consider the homogenous second order system with differential equations

\[
M \frac{d^2x}{dt^2} + f \frac{dx}{dt} + kx = 0
\]
This equation may be written in the standard form
\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = 0 \]
where \( \zeta \) and \( \omega_n \) are the damping factor and undamped natural frequency of the system.

Defining the state variables as \( x = x_1 \) and \( \dot{x} = x_2 \), we get the state equation in the state variable form as
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2\zeta \omega_n x_2 - \omega_n^2 x_1
\end{align*}
\]

These equations may then be solved for phase variables \( x_1 \) and \( x_2 \). The time response plots of \( x_1, x_2 \) for various values of damping with initial conditions can be plotted. When the differential equations describing the dynamics of the system are nonlinear, it is in general not possible to obtain a closed form solution of \( x_1, x_2 \). For example, if the spring force is nonlinear say \((k_1x + k_2x^3)\) the state equation takes the form
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{f}{M} x_2 - \frac{k_1}{M} x_1 - \frac{k_2}{M} x_1^3
\end{align*}
\]

Solving these equations by integration is no more an easy task. In such situations, a graphical method known as the phase-plane method is found to be very helpful. The coordinate plane with axes that correspond to the dependent variable \( x_1 \) and \( x_2 \) is called phase-plane. The curve described by the state point \((x_1,x_2)\) in the phase-plane with respect to time is called a phase trajectory. A phase trajectory can be easily constructed by graphical techniques.

### 7.3.1 Isoclines Method:

Let the state equations for a nonlinear system be in the form
\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*}
\]

Where both \( f_1(x_1, x_2) \) and \( f_2(x_1, x_2) \) are analytic.

From the above equation, the slope of the trajectory is given by
\[
\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = M
\]

Therefore, the locus of constant slope of the trajectory is given by
\[
f_2(x_1, x_2) = Mf_1(x_1, x_2)
\]

The above equation gives the equation to the family of isoclines. For different values of \( M \), the slope of the trajectory, different isoclines can be drawn in the phase plane. Knowing the value of \( M \) on a given isoclines, it is easy to draw line segments on each of these isoclines.

Consider a simple linear system with state equations
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_2 - x_1
\end{align*}
\]

Dividing the above equations we get the slope of the state trajectory in the \( x_1-x_2 \) plane as
For a constant value of this slope say $M$, we get a set of equations

$$\frac{dx_2}{dx_1} = \frac{-x_2 - x_1}{x_2} = M$$

For a constant value of this slope say $M$, we get a set of equations

$$x_2 = \frac{-1}{(M + 1)} x_1$$

which is a straight line in the $x_1$-$x_2$ plane. We can draw different lines in the $x_1$-$x_2$ plane for different values of $M$; called isoclines. If draw sufficiently large number of isoclines to cover the complete state space as shown, we can see how the state trajectories are moving in the state plane. Different trajectories can be drawn from different initial conditions. A large number of such trajectories together form a phase portrait. A few typical trajectories are shown in figure given below.

**Figure 7.1**

The Procedure for construction of the phase trajectories can be summarised as below:

1. For the given nonlinear differential equation, define the state variables as $x_1$ and $x_2$ and obtain the state equations as
   \[
   \dot{x}_1 = x_2 \quad \dot{x}_2 = f(x_1, x_2)
   \]

2. Determine the equation to the isoclines as
   \[
   \frac{dx_2}{dx_1} = \frac{f(x_1, x_2)}{x_2} = M
   \]
3. For typical values of M, draw a large number of isoclines in \( x_1-x_2 \) plane
4. On each of the isoclines, draw small line segments with a slope M.
5. From an initial condition point, draw a trajectory following the line segments with slopes M on each of the isoclines.

**Example 7.1:** For the system having the transfer function \( \frac{1}{z(z+1)} \) and a relay with dead zone as nonlinear element, draw the phase trajectory originating from the initial condition (3,0).

![Diagram](image)

The differential equation for the system is \( \dot{e} = u \) where u is given by

\[
 u = \begin{cases} 
 1 & \text{if } e > 1 \\ 
 0 & \text{if } -1 < e < 1 \\ 
 -1 & \text{if } e < -1 
\end{cases}
\]

Since the input is zero, \( e = r - c = -c \) and the differential equation in terms of error will be

\[ \dot{e} = -u \]

Defining the state variables as \( x_1 = e \) and \( x_2 = \dot{e} \) we get the state equations as

\[
 \begin{align*}
 x_1 &= x_2 \\
 x_2 &= -x_2 - u
\end{align*}
\]

The slope of the trajectory is given by

\[ \frac{dx_1}{dx_2} = \frac{-x_2 - u}{x_2} = M \]

Equation to the isoclines is given by

\[ x_2 = \frac{-x_2 - u}{M} \]

We can identify three regions in the state plane depending on the values of \( e = x_1 \).

**Region 1: \( e = x_1 > 1 \)**

Here \( u = 1 \), so that the isoclines are given by

\[ x_2 = \frac{-e - 1}{M} \quad \text{or} \quad x_2 = \frac{1}{M-1} \]

For different values of M, these are a number of straight lines parallel to the x-axis.

<table>
<thead>
<tr>
<th>M</th>
<th>0</th>
<th>1/3</th>
<th>1/2</th>
<th>2</th>
<th>3</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_2 = \frac{1}{M-1} )</td>
<td>-1</td>
<td>-1.5</td>
<td>-2</td>
<td>1</td>
<td>0.5</td>
<td>-0.2</td>
<td>-0.25</td>
<td>-0.33</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

**Region 2: \( e = -1 < x_1 < 1 \)**
Here \( u = 0 \), so that the isoclines are given by \( x_2 = \frac{-x_1}{M} \) or \( M = -1 \), which are parallel lines having constant slope of -1. Trajectories are lines of constant slope -1.

**Region 3: \( e = x_1 \equiv -1 \)**

Here \( u = -1 \) so that on substitution we get \( x_2 = \frac{-x_1 + 1}{M} \) or \( x_2 = \frac{1}{M+1} \).

These are also lines parallel to \( x - \) axis at a constant distance decided by the value of the slope of the trajectory \( M \).

<table>
<thead>
<tr>
<th>( M )</th>
<th>0</th>
<th>1/3</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_2 )</td>
<td>( \frac{1}{M+1} )</td>
<td>1</td>
<td>0.75</td>
<td>0.5</td>
<td>1/3</td>
<td>0.25</td>
<td>-0.25</td>
<td>-0.33</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

![Isoclines drawn for all three regions are as shown in figure. It is seen that trajectories from either region 1 or 2 approach the boundary between the regions and approach the origin along a line at -1 slope. The state can settle at any value of \( x_1 \) between -1 and +1 as this being a dead zone and no further movement to the origin along the \( x_1 \)-axis will be possible. This will result in a steady state error, the maximum value of which is equal to half the dead zone. However, the presence of dead zone can reduce the oscillations and result in a faster response. In order that the steady state error in the output is reduced, it is better to have as small a dead zone as possible.](image)

**Example 7.2:** For the system having a closed loop transfer function \( \frac{40}{s^2 + 2s + 2} \), plot the phase trajectory originating from the initial condition \((-1,0)\).

The differential equation for the system is given by

\[
\ddot{x} + 2\dot{x} + 5x = 10
\]

Let \( x = x_1 \) and \( \dot{x} = x_2 \).

Then,

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = -2x_2 - 5x_1 + 10
\]
The slope of the trajectory is given by

\[ \frac{d x_2}{d x_1} = \frac{-2x_2 - 5x_1 + 10}{x_2} = M \]

\[ \therefore x_2 = \frac{-5x_1 + 10}{M + 2} \]

When \( x_2 = 0, \)

\[ \frac{-5x_1 + 10}{M + 2} = 0 \quad \text{or} \quad x_1 = 2 \]

ie., all the isoclines will pass through the point \( x_1 = 2, x_2 = 0. \)

When \( M = 0, \)

\[ x_2 = \frac{-5}{2} x_1 + \frac{10}{2} \]

When \( M = 2, \)

\[ x_2 = \frac{-5}{4} x_1 + \frac{10}{4} \]

When \( M = 4, \)

\[ x_2 = \frac{-5}{5} x_1 + \frac{10}{5} \]

When \( M = 8, \)

\[ x_2 = \frac{-5}{10} x_1 + \frac{10}{10} \]

When \( M = -2, \)

\[ x_2 = \infty \]

When \( M = -4, \)

\[ x_2 = \frac{5}{2} x_1 - \frac{10}{2} \]

When \( M = -6, \)

\[ x_2 = \frac{5}{4} x_1 - \frac{10}{4} \]

When \( M = -10, \)

\[ x_2 = \frac{5}{8} x_1 - \frac{10}{8} \]
The isoclines are drawn as shown in figure. The starting point of the trajectory is marked at (-1,0). At (-1,0), the slope is $\infty$, i.e., the trajectory will start at an angle 90°. From the next isoclines, the average slope is \((8+4)/2 = 6\), i.e., a line segment with a slope 6 is drawn (at an angle 80.5°). The same procedure is repeated and the complete phase trajectory will be obtained as shown in figure.

7.3.2 Delta Method:

The delta method of constructing phase trajectories is applied to systems of the form

$$\dot{x} + f(x, \dot{x}, t) = 0$$

Where \(f(x, \dot{x}, t)\) may be linear or nonlinear and may even be time varying but must be continuous and single valued.

With the help of this method, phase trajectory for any system with step or ramp or any time varying input can be conveniently drawn. The method results in considerable time saving when a single or a few phase trajectories are required rather than a complete phase portrait.

While applying the delta method, the above equation is first converted to the form

$$\dot{x} + \omega_n[x + \delta(x, \dot{x}, t)] = 0$$

In general, \(\delta(x, \dot{x}, t)\) depends upon the variables \(x, \dot{x}\) and \(t\), but for short intervals the changes in these variables are negligible. Thus over a short interval, we have

$$\dot{x} + \omega_n[x + \delta] = 0$$

where \(\delta\) is a constant.

Let us choose the state variables as \(x_1 = x; x_2 = \dot{x} / \omega_n\), then

$$\dot{x}_1 = \omega_n x_2$$

$$\dot{x}_2 = -\omega_n(x_1 + \epsilon)$$
Therefore, the slope equation over a short interval is given by

\[ \frac{dx_2}{dx_1} = -\frac{x_1 + \delta}{x_2} \]

With \( \delta \) known at any point \( P \) on the trajectory and assumed constant for a short interval, we can draw a short segment of the trajectory by using the trajectory slope \( dx_2/dx_1 \) given in the above equation. A simple geometrical construction given below can be used for this purpose.

1. From the initial point, calculate the value of \( \delta \).
2. Draw a short arc segment through the initial point with \((-\delta, 0)\) as centre, thereby determining a new point on the trajectory.
3. Repeat the process at the new point and continue.

**Example 7.3:** For the system described by the equation given below, construct the trajectory starting at the initial point \((1, 0)\) using delta method.

\[ \dot{x} + \dot{x} + x^4 = 0 \]

Let \( x = x_1 \) and \( \dot{x} = x_2 \) then

\[ \dot{x}_1 = x_1 \]
\[ \dot{x}_2 = -x_2 - x_1^3 \]

The above equation can be rearranged as

\[ \dot{x}_2 = -(x_1 + x_2 + x_1^3 - x_1) \]

So that \( \delta = x_2 + x_1^3 - x_1 \).

At initial point \( \delta \) is calculated as \( \delta = 0+1-1 = 0 \). Therefore, the initial arc is centred at point \((0, 0)\). The mean value of the coordinates of the two ends of the arc is used to calculate the next value of \( \delta \) and the procedure is continued. By constructing the small arcs in this way the complete trajectory will be obtained as shown in figure.

![Figure 7.4](image-url)
7.4 Limit Cycles:

Limit cycles have a distinct geometric configuration in the phase plane portrait, namely, that of an isolated closed path in the phase plane. A given system may have more than one limit cycle. A limit cycle represents a steady state oscillation, to which or from which all trajectories nearby will converge or diverge. In a nonlinear system, limit cycles describes the amplitude and period of a self sustained oscillation. It should be pointed out that not all closed curves in the phase plane are limit cycles. A phase-plane portrait of a conservative system, in which there is no damping to dissipate energy, is a continuous family of closed curves. Closed curves of this kind are not limit cycles because none of these curves are isolated from one another. Such trajectories always occur as a continuous family, so that there are closed curves in any neighborhood of any particular closed curve. On the other hand, limit cycles are periodic motions exhibited only by nonlinear non conservative systems.

As an example, let us consider the well known Vander Pol’s differential equation

\[
\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0
\]

which describes physical situations in many nonlinear systems.

In terms of the state variables \(x_1 - x\) and \(x_2 - \dot{x}\), we obtain

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \mu(1 - x_1^2)x_2 - x_1
\end{align*}
\]

The figure shows the phase trajectories of the system for \(\mu > 0\) and \(\mu < 0\). In case of \(\mu > 0\) we observe that for large values of \(x_1(0)\), the system response is damped and the amplitude of \(x_1(t)\) decreases till the system state enters the limit cycle as shown by the outer trajectory. On the other hand, if initially \(x_1(0)\) is small, the damping is negative, and hence the amplitude of \(x_1(t)\) increases till the system state enters the limit cycle as shown by the inner trajectory. When \(\mu < 0\), the trajectories moves in the opposite directions as shown in figure.

![Figure 7.5](image)

A limit cycle is called stable if trajectories near the limit cycle, originating from outside or inside, converge to that limit cycle. In this case, the system exhibits a sustained
oscillation with constant amplitude. This is shown in figure (i). The inside of the limit cycle is an unstable region in the sense that trajectories diverge to the limit cycle, and the outside is a stable region in the sense that trajectories converge to the limit cycle.

A limit cycle is called an unstable one if trajectories near it diverge from this limit cycle. In this case, an unstable region surrounds a stable region. If a trajectory starts within the stable region, it converges to a singular point within the limit cycle. If a trajectory starts in the unstable region, it diverges with time to infinity as shown in figure (ii). The inside of an unstable limit cycle is the stable region, and the outside the unstable region.

7.5 Analysis and Classification of Singular Points:

Singular points are points in the state plane where \( \dot{x}_1 = \dot{x}_2 = 0 \). At these points the slope of the trajectory \( dx_2/dx_1 \) is indeterminate. These points can also be the equilibrium points of the nonlinear system depending whether the state trajectories can reach these or not. Consider a linearised second order system represented by

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
\alpha & \beta \\
-\gamma & \delta
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

Using linear transformation \( x = Mz \), the equation can be transformed to canonical form

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
\]

Where, \( \lambda_1 \) and \( \lambda_2 \) are the roots of the characteristic equation of the system.

The transformation given simply transforms the coordinate axes from \( x_1-x_2 \) plane to \( z_1-z_2 \) plane having the same origin, but does not affect the nature of the roots of the characteristic equation. The phase trajectories obtained by using this transformed state equation still carry the same information except that the trajectories may be skewed or stretched along the coordinate axes. In general, the new coordinate axes will not be rectangular.

The solution to the state equation being given by

\[
\begin{align*}
z_1(t) &= e^{\lambda_1 t}z_1(0) \\
z_2(t) &= e^{\lambda_2 t}z_2(0)
\end{align*}
\]

The slope of the trajectory in the \( z_1-z_2 \) plane is given by

\[
\begin{align*}
\frac{d\dot{z}_2}{d\dot{z}_1} &= \frac{\dot{z}_2}{\dot{z}_1} = \tan \theta \\
\frac{d\dot{z}_2}{d\dot{z}_1} &= \frac{\dot{z}_2}{\dot{z}_1} = \frac{\lambda_2}{\lambda_1} \frac{d\dot{z}_1}{d\dot{z}_1}
\end{align*}
\]

\[ \ln(z_2) = \left(\frac{\lambda_2}{\lambda_1}\right) \ln(z_1) \quad \text{or} \quad z_2 = c(z_1)^{\lambda_2/\lambda_1} \]

Based on the nature of these eigen values and the trajectory in \( z_1-z_2 \) plane, the singular points are classified as follows.

**Nodal Point:**

Consider eigen values are real, distinct and negative as shown in figure (a). For this case the equation of the phase trajectory follows as \( z_2 = c(z_1)^{\lambda_2} \).
Where, \( k_1 = (\lambda_2/\lambda_1) \geq 0 \) so that the trajectories become a set of parabola as shown in figure (b) and the equilibrium point is called a node. In the original system of coordinates, these trajectories appear to be skewed as shown in figure (c).

If the eigen values are both positive, the nature of the trajectories does not change, except that the trajectories diverge out from the equilibrium point as both \( z_1(t) \) and \( z_2(t) \) are increasing exponentially. The phase trajectories in the \( x_1-x_2 \) plane are as shown in figure (d). This type of singularity is identified as a node, but it is an unstable node as the trajectories diverge from the equilibrium point.

**Saddle Point:**
Consider now a system with eigen values are real, distinct one positive and one negative. Here, one of the states corresponding to the negative eigen value converges and the one corresponding to positive eigen value diverges so that the trajectories are given by \( z_2 = c(z_1)^k \) or \( (z_1)^k z_2 = c \) which is an equation to a rectangular hyperbola for positive values of \( k \). The location of the eigen values, the phase portrait in \( z_1-z_2 \) plane and in the \( x_1-x_2 \) plane are as shown in figure. The equilibrium point around which the trajectories are of this type is called a saddle point.

**Focus Point:**
Consider a system with complex conjugate eigen values. The canonical form of the state equation can be written as

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix}
\sigma + j\omega & 0 \\
0 & \sigma - j\omega
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
\]

Using linear transformation, the equation becomes

**Figure 7.7**
\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} = \begin{bmatrix}
\sigma & \omega \\
-\omega & \sigma
\end{bmatrix} \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
\]

The slope
\[
\frac{dy_2}{dy_1} = \frac{-\omega y_1 + \sigma y_2}{\sigma y_1 + \omega y_2} = \frac{y_2 - ky_1}{y_1 - ky_2}
\]

Define \(\frac{dy_2}{dy_1} = \tan \psi\) and \(\frac{y_2}{y_1} = \tan \theta\)

We get,
\[
\tan \psi = \frac{\tan \theta - k}{1 + k \tan \theta}\quad \text{or} \quad \tan(\theta - \psi) = k
\]

This is an equation for a spiral in the polar coordinates. A plot of this equation for negative values of real part is a family of equiangular spirals. The origin which is a singular point in this case is called a stable focus. When the eigen values are complex conjugate with positive real parts, the phase portrait consists of expanding spirals as shown in figure and the singular point is an unstable focus. When transformed into the \(x_1-x_2\) plane, the phase portrait in the above two cases is essentially spiralling in nature, except that the spirals are now somewhat twisted in shape.

**Figure 7.8**

Centre or Vortex Point:
Consider now the case of complex conjugate eigen values with zero real parts.

i.e., \(\lambda_1, \lambda_2 = \pm j\omega\)
**Integrating the above equation, we get** \( y_1^2 + y_2^2 = R^2 \) **which is an equation to a circle of radius** \( R \). The radius \( R \) can be evaluated from the initial conditions. **The trajectories are thus concentric circles in** \( y_1-y_2 \) **plane and ellipses in the** \( x_1-x_2 \) **plane as shown in figure. Such a singular points, around which the state trajectories are concentric circles or ellipses, are called a centre or vortex.**

![Figure 7.9](image)

**Example 7.4:**

Determine the kind of singularity for the following differential equation.

\( \ddot{y} + 3\dot{y} + 2y = 0 \)

Let the state variables be \( x_1 = y \) **and** \( x_2 = \dot{y} \)

The corresponding state model is

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -3x_2 - 2x_1
\end{align*}
\]

At singular points, \( \dot{x}_1 = x_2 = 0 \) **and** \( \dot{x}_2 = -3x_2 - 2x_1 = 0 \)

Therefore, the singular point is at (0,0)

\[
A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}
\]

The characteristic equation is \(|\lambda I - A| = 0\)

\[
i.e., \begin{vmatrix} \lambda & -1 \\ -2 & -3 \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 + 3\lambda + 2 = 0
\]

\( \lambda_1, \lambda_2 = -2, -1 \). **Since the roots are real and negative, the type of singular point is stable node.**

**Example 7.5:** For the nonlinear system having differential equation:

\( \ddot{y} - \left(0.1 - \frac{10}{3} \dot{y}^2\right)y + y^2 = 0 \), find all singularities.

Defining the state variables as \( y = x_1 \), \( \dot{y} = x_2 \), the state equations are

\( \dot{x}_1 = x_2 \)
\( \dot{x}_2 = -x_1 + 0.1x_2 - x_1^2 - \frac{10}{3}x_2^3 \)

At singular points, \( \dot{x}_1 = x_2 = 0 \) and \( \dot{x}_2 = -x_1 + 0.1x_2 - x_1^2 - \frac{10}{3}x_2^3 = 0 \)

So that the singular points \( x_2 = 0 \) and \( x_1 = 0 \), i.e., \( x_1(x_1 + 1) = 0 \)

The singularities are thus at (0,0) and (-1,0).

**Linearization about the singularities:**

The Jacobean matrix \( J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(1 + 2x_1) & 0.1 - 10x_2^2 \end{bmatrix} \)

Linearization around (0,0), i.e., substituting \( x_1 = 0 \) and \( x_2 = 0 \)

\( J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0.1 \end{bmatrix} \)

The characteristic equation is \( |\lambda I - A| = 0 \)

\[ \begin{vmatrix} \lambda & -1 \\ 1 & \lambda - 0.1 \end{vmatrix} = 0 \]

i.e., \( \lambda(\lambda - 0.1) + 1 = 0 \) or \( \lambda^2 - 0.1\lambda + 1 = 0 \)

\( \lambda_1, \lambda_2 = 0.5(0.1 \pm j\sqrt{3.99}) \)

The eigen values are complex with positive real part. The singular point is an unstable focus.

**Linearization around (-1,0)**

\( J(-1,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0.1 \end{bmatrix} \)

The characteristic equation will be

\[ \begin{vmatrix} \lambda & -1 \\ -1 & \lambda - 0.1 \end{vmatrix} = 0 \]

Therefore \( \lambda^2 - 0.1\lambda - 1 = 0 \)

\( \lambda_1, \lambda_2 = 1.05 \) and \(-0.98\). Since the roots are real and one negative and another positive, the singular point is a saddle point.

**Example 7.6:**

Determine the kind of singularity for the following differential equation.

\( \ddot{x} + 0.5\dot{x} + 2x + x^2 = 0 \)

Let the state variables be \( x_1 = x \) and \( x_2 = \dot{x} \)

The corresponding state model is

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -0.5x_2 - 2x_1 - x_1^2
\end{align*} \]
At singular points, $\dot{x}_1 = x_2 = 0$ and $\dot{x}_2 = -0.5x_2 - 2x_1 - x_1^2 = 0$.
So that the singular points $x_2 = 0$ and $-2x_1 - x_1^2 = 0$. i.e., $x_1(x_1 + 2) = 0$

The singularities are thus at (0,0) and (-2,0).

Linearization about the singularities:

The Jacobean matrix $J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$

Linearization around (0,0), i.e., substituting $x_1 = 0$ and $x_2 = 0$

$J(0,0) = \begin{bmatrix} 0 & 1 \\ -2 & -0.5 \end{bmatrix}$

The characteristic equation is $|\lambda I - A| = 0$

$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 0.5 \end{vmatrix} = 0$

i.e., $\lambda(\lambda + 0.5) + 2 = 0$ or $\lambda^2 + 0.5\lambda + 2 = 0$

$\lambda_1, \lambda_2 = (-0.25 \mp j1.39)$

The eigen values are complex with negative real parts. The singular point is a stable focus.

Linearization around (-2,0)

$J(-2,0) = \begin{bmatrix} 0 & 1 \\ 2 & -0.5 \end{bmatrix}$

The characteristic equation will be

$|\lambda & -1 \\ -2 & \lambda + 0.5| = 0$

Therefore $\lambda^2 + 0.5\lambda - 2 = 0$

$\lambda_1, \lambda_2 = 1.19$ and -1.69. Since the roots are real and one negative and another positive, the singular point is a saddle point.